Bethe lattice representation for sandpiles

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Avalanches in sandpiles are represented by a process of percolation in a Bethe lattice with a feedback mechanism. The results indicate that the frequency spectrum and probability distribution of avalanches provide a better resemblance to the experimental results than other models using cellular automata simulations. Apparent discrepancies between experiments performed by different authors are reconciled. Critical behavior is expressed here by the critical properties of percolation phenomena. [S1063-651X(99)08306-3]

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I. INTRODUCTION

The idea of self-organized criticality (SOC) proposed by Bak, Tang, and Wiesenfeld [1,2] has triggered a lot of experimental as well as theoretical work on relaxation processes in granular materials. Sandpiles seem to be the simplest systems to test self-organized behavior. The importance of its study is due to the fact that SOC has been suggested as a possible explanation for the power law behavior seen in many systems: earthquakes [3], mass distribution in the Universe, star flickers, etc. [4]

Experiments on sandpiles have been designed and performed by various authors [5,6]. Avalanche sizes have been recorded in rotating drum experiments [5] and it has been found that avalanches occurred quite regularly in size and time, in an almost periodic pattern [7,8], instead of being distributed over all sizes obeying a power law distribution as was predicted in [1,2].

Mass fluctuations in an evolving sandpile have been studied [6], showing that for sufficiently small sandpiles, the observed mass fluctuations are scale invariant and probability distribution of avalanches shows finite size scaling whereas large sandpiles do not. In this experiment, small sandpiles seemed to exhibit SOC. In addition, an apparent disagreement has emerged between the results reported in [5] and [6] but, as we will show in this paper, these results are essentially the same and no contradiction exists.

Though many other theoretical and experimental works have been carried out [9–28], some of the more recently proposed models have been devoted to the problem of SOC in a more general fashion rather than the sole application to sandpiles (e.g., [9,10,12,16]). Other studies restrict their attention to models for which particular mechanisms of interaction seem to be relevant [13–15,18,21,28].

In the present work we propose a representation of the avalanche process in sandpiles as a percolation in a Bethe lattice, capturing the essential features of the avalanche phenomenon and simultaneously taking into account the nature of the sandpile as a granular medium, in order to reproduce the experimental results. The reason for this representation is mainly intuitive: The image of an avalanche as an initial object that consecutively drags another resembles a branching process for which the Bethe lattice representation seems to be natural.

This branching process has been proposed previously in [22] and has proved to be an adequate method to describe the change of behavior of fragment size distribution in fragmentation phenomena. Another branching process representation has been proposed [12,26] in an attempt to obtain analytical solutions for avalanche processes. Other authors present a self-organizing branching process [23,24]. However, the branching structure is not related to the physical nature of the system, although a feedback mechanism is introduced.

In our work, the process of dragging is characterized by a *drag probability* p for each one of the particles forming the nodes of the lattice. The nature of the system is taken into account through the relation of p with the parameters characterizing the self-organizing characteristics of the system, i.e., the slope angle θ and the size (number of grains) of the pile N. This viewpoint is similar to that of other authors [23,24], but with a much closer relation with the experiments and the physical nature of the studied systems, i.e., sandpiles. The main features obtained there can be reproduced with this representation, but we will focus in the present study on the polemics related with experimental data [7].

In Sec. II we describe the representation of the avalanches and expose the relation between the drag probability and sandpile characteristics. In Sec. III, the results of simulations with this representation are exposed and compared with experimental results. Section IV is devoted to the conclusions.

II. REPRESENTING THE SANDPILE AND THE AVALANCHES

Let us represent the avalanche as a cascade process in the Bethe lattice as follows. First, we start with a single node, which could represent in this case a grain. With probability p, from this node F new nodes will emerge, representing that the initial particle has generated $F-1$ new indistinguishable particles, which, in principle, will continue the avalanche together with the initial one. This operation (generation of new identical particles with probability p) is applied to each node of this new group. In this case some of them will continue the generation, some will not, and so on. The process is represented in Fig. 1 for $F=3$. Empty nodes represent those in which the process of percolation in this lattice will not progress, mimicking those particles that do not follow the

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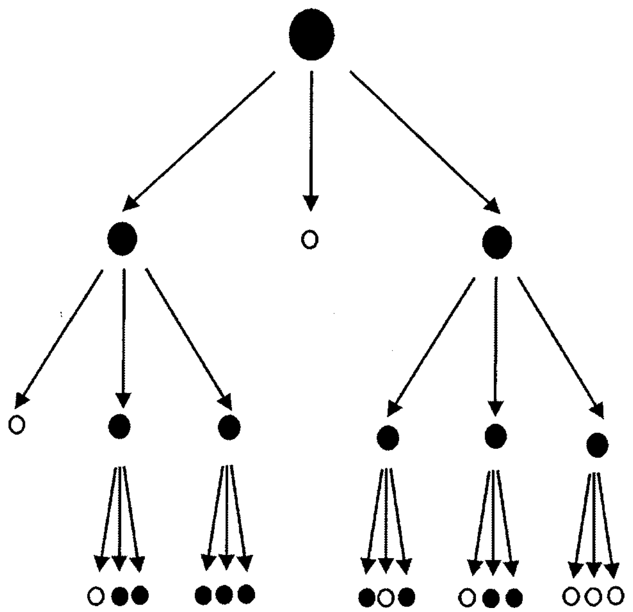

FIG. 1. Bethe lattice representation of the sandpile behavior.

falling process in the cascade. This representation seems to be natural for the process and does not appeal to the nature of the forces between the grains in the sandpile, the nature of the grains, or even the nature of the avalanche.

Once the percolation process exceeds a given length (that of the border of the sandpile), those nodes beyond the limit constitute the avalanche. Counting the number of grains that just surpass the length of the pile is equivalent to measuring the size of the avalanche. Knowing the number of grains in the pile, it is possible to register the mass variations in the pile with time $M(t)$. The percolation process stops once the limit is surpassed; the pile *reorganizes itself* with the new number of grains (i.e., a new slope is calculated with the remaining grains) and again commences to grow until a new avalanche develops. In this model this establishes a feedback mechanism that can be related to the experimental conditions.

We now relate the percolation probability with the parameters characterizing the pile, namely, its slope and size, using the simplest representation. To do this let us represent a conic sandpile of height h, radius R, and base slope θ. A small sphere of "effective radius" r_0 can characterize the size of the grain of sand. The ratio of the volume of the pile to that of the grain (including porosity effects in the value of r_0) gives the number of grains in the pile

$$N=\frac{1}{4}x^3\tan\theta,$$

where $x=R/r_0$ is the ratio of the radius of the pile and the effective radius of the grain.

The percolation probability is in this case translated to a *drag probability* $p(\theta)$ of one grain to the next $F-1$ situated down the slope. In this way, the interpretation of Fig. 1 is straightforward.

The dependence of p on θ can be formulated *a priori* by taking into account that the drag probability should be an increasing function of the slope. A good variable to describe this slope seems to be $\tan\theta$. On the other hand, the drag probability should be small for angles less than the critical angle and once that angle is surpassed, the probability for an avalanche to take place must increase sharply. Let us propose the dependence

$$p(\theta,T)=\frac{\exp\left(\frac{\tan\theta-\tan\theta_c}{T}\right)}{\exp\left(\frac{\tan\theta-\tan\theta_c}{T}\right)+1}, \tag{1}$$

where θ_c is the critical angle, related to Coulomb's law $\mu=\tan\theta_c$ (μ is the friction coefficient and here, for simplicity, will be taken as unity), and T is a parameter through which we may control the sharpness of the variation just at $\theta=\theta_c$ and can be used to include factors such as granularity and vibrations [27]. Using the relation between the slope angle and the number of grains in the pile, in this case Eq. (1) can be expressed as

$$p(y,T)=\frac{1}{1+\exp\left(\frac{1-y}{T}\right)}, \tag{2}$$

where $y=N/N_c$. N_c is the number of grains corresponding to θ_c. This is valid for small T so that the variation of $p(y,T)$ is sharp near $y=1$, i.e., $T\ll 1/4$.

Once the pile reaches a size of approximately N_c the avalanches will be noticeable and the slope will be readjusted after each avalanche. The process of adding grains will again vary the value of p up to values near p_c, the value of the probability corresponding to θ_c, to produce another avalanche and so on. This constitutes a mechanism of feedback since the flux of sand tends to remain constant because of the concurrence of sand supply and avalanches. This has been recognized as necessary [25] to deal with critical behavior. Because of size effects, avalanches will be registered for p slightly less than p_c.

If, for a given value of p, an avalanche develops, it will be counted if the number of steps in the Bethe lattice surpasses the length of the slope of the pile. The length of the slope L measured in units of the grain diameter is $g=x/2\sqrt{1-p}$, which is the threshold for an avalanche to be registered. When the value g is surpassed, those grains (nodes in the Bethe lattice) belonging to that generation are counted as the size of the avalanche. This will permit the relation of the results of the simulations with the measured magnitudes, namely, the size of the avalanches $S(t)$ [5] or the mass of the piles $M(t)$ [6] as a function of time. In this last case, the mass of the sandpile is represented as the number of grains $N(t)$.

III. RESULTS OF THE SIMULATIONS

Simulations have been performed for a wide range of values of x, ranging from 10 to 500, using Eq. (2) for $T=0.1$ and a Bethe lattice with two branches. For each value of x, more than $2^{16}\times 100$ realizations were performed. Collected data were $N(t)$ and $S(t)$.

The temporal fluctuations of the mass $N(t)$, measured as the number of grains in the pile, in units of N_c, are plotted in

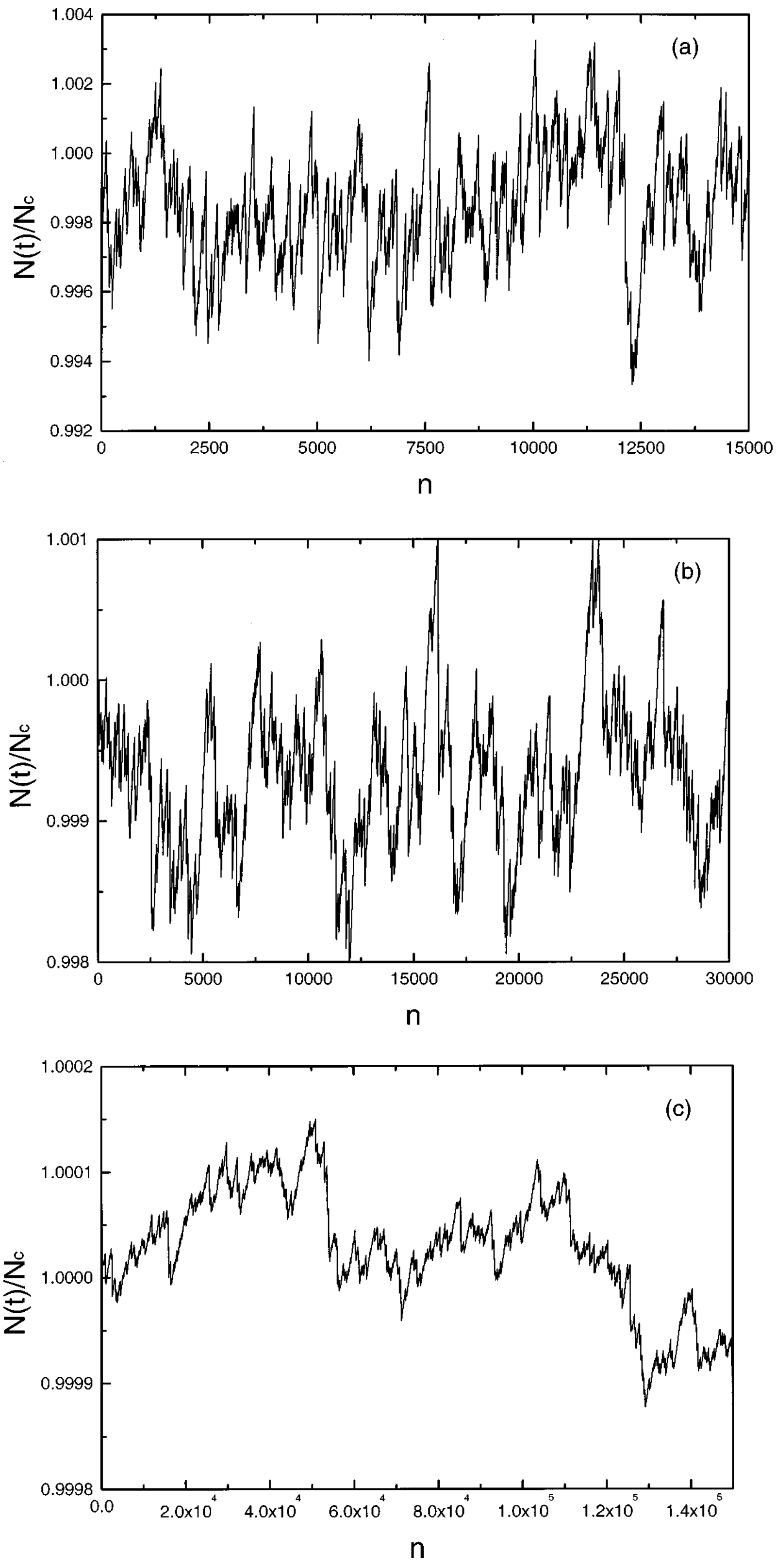

FIG. 2. Fluctuations of the mass of the sandpile $N(t)/N_c$ for (a) $x=50$, (b) $x=100$, and (c) $x=500$ obtained from simulations in a Bethe lattice as described in the text. Here t is a dimensionless quantity as described in the text. The behavior for other number of terminals in the lattice or another plausible dependence of $p(\theta)$ is essentially the same.

Figs. 2(a), 2(b), and 2(c) for $x=50$, 100, and 500, respectively. The unit of time used was that between two consecutive events of adding grains. Avalanches are considered to be instantaneous. This behavior resembles that reported in [6]. Note that different time scales were used in Figs. 2(a)–2(c) for a better illustration of the time variation for different sizes.

Figure 3 shows the probability of avalanche sizes $P(s)$ scaled as $x^{1.9}$, as in [6], vs s normalized to $x^{0.95}$ for $x=50$ and 500, showing a good finite size scaling $P(s)x^{\beta}$ vs s/x^{ν} as in [6]. The system shows finite size scaling

$$P(s) \sim x^{-\beta} f(s/x^{\nu}), \tag{3}$$

$f(x)$ being a scaling function such that $f(x) \simeq \text{const}$ for $x \ll 1$ and $f(x) \to 0$ for $x \gg 1$. The drop number $\langle s \rangle = 1$ implies

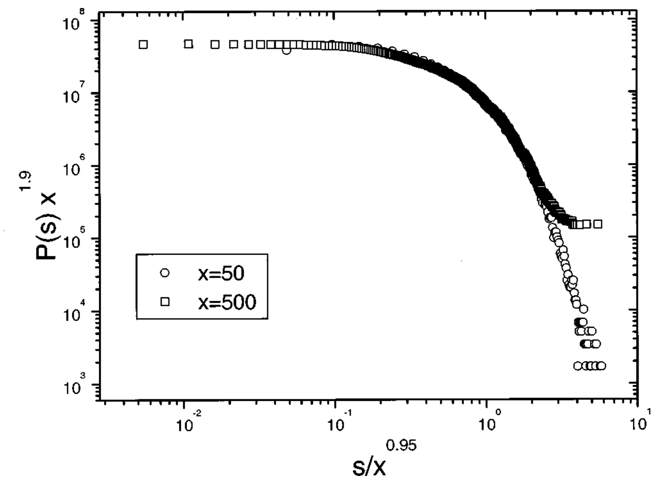

FIG. 3. Avalanche probability scaled as $x^{1.9}$ vs avalanche size s normalized to $x^{0.95}$ for $x=50$ (circles) and 500 (squares). The result for this theoretical sandpile resemble those of [6] with exponential falloff. Large piles show larger dispersion for large avalanches.

that $\beta=2\nu$ [9] since the pile is in a steady state, i.e., on average one grain must fall for each grain added.

The exponent $\nu=0.95$ was chosen as the best fitting for all data with $s/x^{\nu}<2$. This representation reproduces the finite size scaling for all sizes of sandpiles. The exponential falloff can be verified.

Larger piles show more dispersion for large avalanche sizes. This can be explained noting that in the percolation process, larger percolations correspond to probabilities near the critical one, where critical behavior dominates and fluctuations are stronger.

Figures 4(a) and 4(b) show the power spectrum of $N(t)$ and $s(t)$, respectively, for $x=50$, 100, and 500 exhibiting in Fig. 4(a) a clear $1/f^2$ dependence, which also coincides with the results of [6]. The power spectrum in Fig. 4(a) reveals the same characteristics for all sandpiles, i.e., the dependence of $1/f^{\beta}$ on $\beta\simeq 2$, whereas in Fig. 4(b) the power spectrum is clearly flat as in [5].

Concerning the apparent disagreement between the experimental results in [5] and [6], it must be said that the process of measurement in both experiments is essentially different since in [6] the mass of the pile is recorded as a function of time, whereas in [5] the experiments record the variation of avalanche size as a function of time, i.e., the magnitude that would correspond to the temporal variation of the number of grains in the rotating cylinder, which means its time derivative. This leads to a different power spectrum, so that if the power spectrum obtained in [6] is $1/f^2$, in this case its derivative should have a ''flat'' spectrum, in corre-

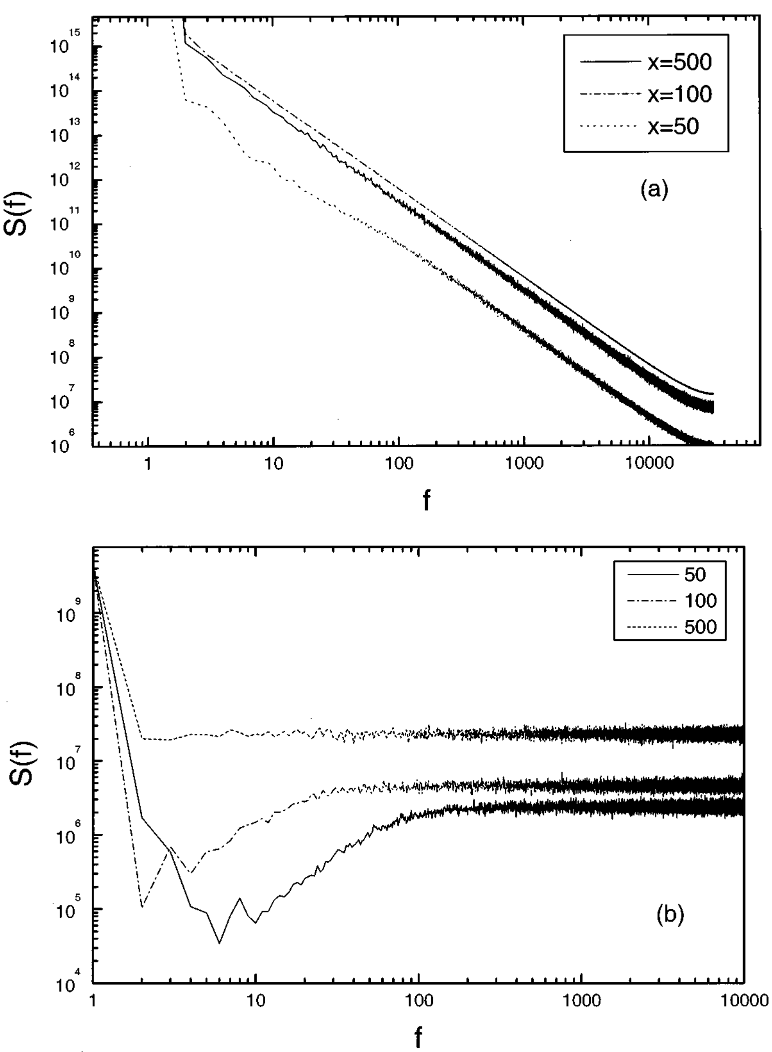

FIG. 4. (a) Power spectrum of mass fluctuations for different sizes of the sandpiles $x=50$, 100, and 500 as indicated in the figure. The spectrum is $1/f^2$ in agreement with [6]. The corresponding spectrum for the avalanches is flat as in [5]. (b) Power spectrum of the avalanche sizes for the same set of values as in (a).

spondence with the results of [5], and there is no disagreement.

In our case, the spectrum is flat for high frequencies because we are considering the avalanches as instantaneous, but this is not an essential point. The simulations could be improved introducing a finite time for avalanches. Though both teams have argued about the differences of their experimental setups, we think that our argument shows a very important difference. The sizes of the avalanches $S(t)$ represent the value of the variation of the function $N(t)$ in each jump, i.e., the derivative of that function, so that the spectra in [5] and [6], though they could be interpreted as expressions of different behavior, are really intimately related.

The particular dependence of the drag probability with the slope is not of great importance for the main results of this work. Dependences on $\sin\theta$, $\sin^2\theta$, and others can be used in the simulations without significant changes in the results concerning probability distribution of avalanches, power spectrum, etc. The main property required is the increase of the drag probability with the slope. Also, the number of terminals in the Bethe lattice is not important for the main conclusions. This reveals the robustness of this phenomenon.

IV. CONCLUSIONS

A Bethe lattice representation linked with characteristics of sandpiles, including a feedback mechanism, has been presented in the same direction outlined in [23] with a different viewpoint, closely related to the physical nature of the sandpile, which leads us to a closer link with experimental results. The reproducibility of the experimental results is based on the fact that the Bethe lattice representation unravels the essence of the avalanche process in sandpiles and is able to be linked with the geometrical properties of the system. Other types of representations, such as cellular automata, make the introduction of the physical characteristics of sandpiles more difficult.

This representation keeps the same nature for all avalanches, irrespective of pile size. In the avalanche process there seems to exist some kind of transition, manifested in the change of behavior of the size distribution of avalanches when the piles are large, reflected in the increase of fluctuations in the region of large avalanches. This is related to the process of percolation in the Bethe lattice near the critical point. The nature of the phenomenon as a second-order phase transition is present here as a percolation phenomenon.

In this way, the description of avalanches has been translated to the problem of percolation in a Bethe lattice and in this sense the phenomenon is critical. Thus SOC, examined from this viewpoint, is present in the organization of avalanches.

The proposed representation is intuitively very appealing, seems to reproduce the behavior of the sandpiles, is very simple to implement even in a small computer, and reveals an essentially unique behavior in small and large sandpiles. In addition, it contains the main characteristics of sandpiles in the sense of the increase of avalanche probability by adding sand grains and a readjustment of the slope after each avalanche.

This representation works without the inclusion of a detailed interaction between the grains or a detailed description of their geometry. It is a mean field viewpoint since the interaction between branches is neglected.

Oscillations of p and θ near a critical value are properties of this model as they are also in the branching process model proposed in [23,24]. Finite size scaling for different sizes was obtained for the distribution of avalanches with good reproduction of the experimental behavior. Another difference of our representation from cellular automata is that a local toppling rule is not needed here and only a law of variation of the drag probability with the slope is needed for this model, this being a global property of the system, i.e., it is related to the total number N of grains in the pile.

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